U.G. 5th Semester Examination - 2020

MATHEMATICS

Course Code: BMTMCCHT 501

Course Title: Algebra-III

Full Marks : 40 Time : 2 Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meanings.

1. Answer any **ten** questions: $1\times10=10$

a) Find the correct answer:

The order of the quotient group $\mathbb{Z}/10\mathbb{Z}$ is–

i) 1 ii) 5

iii) 9 iv) 10

b) Consider the quotient group $\mathbb{Q}/\mathbb{Z}$ of the additive group of rational numbers. Find the order of the element $\frac{2}{3}+\mathbb{Z}$ in $Q/\mathbb{Z}$.

c) Is A_3 a normal subgroup of S_3?

d) Find the correct answer:

The number of group homomorphisms from the cyclic group $\mathbb{Z}_4$ to the cyclic group $\mathbb{Z}_7$ is–

i) 4 ii) 7

iii) 1 iv) 3

e) Give an example of an infinite non-commutative group.

f) Find the correct answer:

Let F be a field. Then the number of ideals of F is–

i) 0 ii) 1

iii) 2 iv) infinite

g) Consider the ideal $3\mathbb{Z}$ in the ring $\mathbb{Z}$. Is $3\mathbb{Z}$ a prime ideal of $\mathbb{Z}$?

h) Let F be a field. Find a maximal ideal of F.

i) For a linear transformation $T:V\to W$, where V and W are vector spaces over the field F, show that $T(x-y)=T(x)-T(y)$ for all $x, y\in V$.

j) Let $T:\mathbb{R}^2(\mathbb{R})\to\mathbb{R}^2(\mathbb{R})$ be defined by $T(a_1, a_2)=(a_1, -a_2)$ for all $(a_1, a_2)\in\mathbb{R}^2$. Is T linear? Justify.

k) Does there exists any one-one linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$?

l) Find the eigenvector(s) corresponding to the eigenvalue -1 of the matrix $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$.

m) State Cayley-Hamilton theorem for a square matrix.

n) Find the associated matrix of the real quadratic form $x_1^2 - x_2^2 + 2x_3^2$ in three variables x_1, x_2, x_3.

o) Prove that $\langle x, 0 \rangle = O$.

2. Answer any **five** questions: $2\times5=10$

a) Suppose G be a group and a map $\phi : G \to G$ is defined by $\phi(x) = x^2$, $x \in G$. Prove that ϕ is a homomorphism if G is commutative.

b) Show that $(Q, +)$ is not isomorphic to $(Q^+, \cdot)$.

c) Consider the direct product $\mathbb{Z} \times \mathbb{Z}$. Is it a cyclic group? Justify.

d) Why is the ideal $6\mathbb{Z}$ not a maximal ideal of $\mathbb{Z}$?

e) Prove that the fields $\mathbb{R}$ and $\mathbb{C}$ are not isomorphic.

f) Find two linear operators T and U on $\mathbb{R}^2$ s.t. $TU=0$ but $UT \neq 0$.

g) If λ be an eigenvalue of a non-singular matrix A, then show that λ^{-1} is an eigenvalue of A^{-1}.

h) Consider the matrix $A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix}$. Is A diagonalisable?

3. Answer any **two** questions: $5\times2=10$

a) i) Let G be a group and let Z(G) be the center of G. If G/Z(G) is a cyclic group, prove that G is abelian.

ii) Show that if $\phi : R \to S$ is a ring homomorphism then $\operatorname{Ker}\phi$ is an ideal of R. $3+2=5$

b) i) Prove that a commutative ring with unity having no non-trivial proper ideals is a field.

ii) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $T(x, y) = (ax+by,\ cx+dy)$, $(x, y) \in \mathbb{R}^2$. If $ad-bc \neq 0$ then find the dimension of Ker T. $3+2=5$

c) i) Let V and W be two vector spaces over the field Q of rational numbers and let $T: V \to W$ be a function such that $T(x+y) = T(x) + T(y)$ for all $x, y \in V$. Then show that $T(cx) = ct(x)$ for all $c \in Q$ and $x \in V$.

ii) Prove that the eigenvalues of a real skew symmetric matrix is either zero or purely imaginary. 3+2=5

4. Answer any **one** question: $10\times1=10$

a) i) Prove that in a commutative ring R with unity. An ideal I is a prime ideal if and only if the quotient ring R/I is an integral domain.

ii) Prove that the symmetric group S_3 has a trivial center.

iii) Let $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$ be ordered bases of the real vector spaces V and W respectively. A linear transformation $T: V \to W$ maps the basis vectors as $T(\alpha_1) = \beta_1$, $T(\alpha_2) = \beta_1 + \beta_2$, $T(\alpha_3) = \beta_1 + \beta_2 + \beta_3$. Show that T is non-singular. 4+3+3=10

b) i) State and prove the fundamental theorem of group homomorphism.

ii) Reduce the quadratic form $5x^2 + y^2 + 10z^2 - 4yz - 10zx$ to the normal form and show that it is positive definite. 5+5=10

c) i) Prove that up to isomorphic there is only one infinite cyclic group.

ii) Prove that each eigenvalue of a real orthogonal matrix has unit modulus. 5+5=10

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