U.G. 2nd Semester Examination - 2021

MATHEMATICS

[HONOURS]

Course Code : BMTMCCHT201

Course Title : Real Analysis-I

Full Marks : 40 Time : 2 Hours

The figures in the right-hand margin indicate marks.

Notations and Symbols have their usual meanings.

1. Answer any **ten** questions: 1×10=10

a) What is the law of trichotomy in $\mathbb{R}$?

b) Show that $\lim_{n\to\infty}\frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{n}\right)=0$.

c) If $a, b \in \mathbb{R}$ and $0 \le a-b < \varepsilon$ holds for every positive ε, then prove that a = b.

d) Show that the sequence $\left\{(-1)^n\right\}$ does not converges.

e) Give an example to show that intersection of an infinite number of neighbourhoods of a point may not be a neighbourhood of that point.

f) Give an example of a set with only $\sqrt{2}$ as a limit point.

g) State Bolzano - Weierstrass theorem for an infinite subset of $\mathbb{R}$.

h) State Cauchy's condensation test for the series.

i) Give an example of a set $S \subsetneq \mathbb{R}$ such that S is neither open nor closed in $\mathbb{R}$.

j) Prove or disprove that the every finite set is open.

k) Give an example of a sequence of irrational numbers that converges to a rational number.

l) If c is a constant and $\sum a_n$, converges to a, then show that $\sum ca_n$ converges to ca.

m) Find $\sup\left\{1,\frac{1}{2},\frac{1}{3},\ldots\right\}$ and $\inf\left\{1,\frac{1}{2},\frac{1}{3},\ldots\right\}$.

n) If $\{a_n\}$ is a bounded sequence, then show that $\overline{\lim_{n\to\infty}}(-a_n)=-\underline{\lim_{n\to\infty}}\, a_n$.

o) Prove that the series $\sum_{n=1}^{\infty} U_n$, where $U_n=\frac{n}{n+1}$ is divergent.

2. Answer any **five** questions: 2×5=10

a) Give an example of an ordered field which does not have the supremum property (completeness). Justify.

b) If $\sum a_n^2$ and $\sum b_n^2$ are both convergent series, prove that the series $\sum a_n b_n$ is also convergent.

c) Give an example of a sequence which is bounded below but unbounded above.

d) Show that for any real number x, there exist a unique integer m such that $m \le x < m+1$.

e) Prove that

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^v+1}}+\frac{1}{\sqrt{n^v+2}}+\ldots+\frac{1}{\sqrt{n^v+n}}\right]=1.$$

f) Prove that every bounded sequence in R contains a convergent subsequence.

g) Show that the series

$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots$, is convergent and find the sum of the series.

h) Define conditionally convergent series with example.

3. Answer any **two** questions: 5×2=10

a) i) Prove that the set of all, positive rational numbers is countable.

ii) Show that Cauchy's root test establishes the convergence of the series $\sum_{n=1}^{4} 3^{-n-(-1)^n}$ while D'Alembert's ratio test fails to do so. 3+2

b) Prove that the sets

$S=\{x \in \mathbb{R} : 2x^2-5x+2<0\}$ and

$T=\{x \in \mathbb{R} : 2x^2-5x+2>0\}$

are open in $\mathbb{R}$. 5

c) i) Use Leibnitz test to show that $\sum_{n=1}^{4}\frac{(-1)^n(n+5)}{n(n+1)}$ is convergent.

ii) Find the derived set of

$S=\left\{m+\frac{1}{n}; m \in N,\ n \in N\right\}$. 3+2

4. Answer any **one** question: 10×1=10

a) i) If A is countable and B is countable then prove that $A \cup B$ is countable.

ii) If $\{a_n\}$ is a null sequence and $\{b_n\}$ is a bounded sequence, then show that $\{a_n b_n\}$ is a null sequence.

iii) Give an example of a sequence $\{a_n\}$ which is not bounded but for which $\lim_{n\to\infty} \frac{a_n}{n} = 0$. 5+3+2

b) i) Prove that the sequence $\{x_n\}$ defined by $x_1 = \sqrt{5}, x_{n+1} = \sqrt{5 + x_n}$ for all $n \geq 1$, converges to the positive root of the equation $x^2 - x - 5 = 0$. 5

ii) State Cauchy's general principle of convergence to show that the sequence $\{x_n\}$ where $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n}$ is divergent. 2+3

c) i) Examine the convergence of the series

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots, x > 0$$ 3

ii) Prove that every absolutely convergent series is convergent. 2

iii) Test the convergence of the series

$$1 + \frac{1}{2} \cdot \frac{1}{3} + \frac{1.3}{2.4} \cdot \frac{1}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{1}{7} + \ldots$$

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